Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 11

Due date: July 7th, 2026.

Problem 31 Let R be a ring. The goal of this exercise is to prove that the category (R-Mod) of R-modules has enough injective objects.

i) For an R-module M define an R-module structure on the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We denote the resulting R-module by $\widehat{M}$.

ii) Show that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

iii) Use ii) to prove that $\widehat{R}$ is an injective R-module.

iv) Show that arbitrary products of injective objects are injective.

v) Let M be an R-module. Choose a surjection from a free module $R^I \twoheadrightarrow \widehat{M}$. Construct an injective map $M \hookrightarrow (\widehat{R})^I$.

vi) Infer that (R-Mod) has enough injective objects.

Problem 32 Let $f : X \to Y$ be an affine morphism of schemes, i.e., for some open affine cover $Y = \bigcup_i V_i$ the preimages $f^{-1}(V_i)$ are affine. Prove that

$$f_* : \{\text{Quasi-coherent } \mathscr{O}_X\text{-modules}\} \longrightarrow \{\text{Quasi-coherent } \mathscr{O}_Y\text{-modules}\}$$

is an exact functor.

Hint. Recall that whenever $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes, then

$$f_* : \{\text{Quasi-coherent } \mathscr{O}_B\text{-modules}\} \longrightarrow \{\text{Quasi-coherent } \mathscr{O}_A\text{-modules}\}$$

is an exact functor.

Problem 33 (Riemann–Roch) Let k be an algebraically closed field and C be a smooth proper connected curve over k. Assume that there exists a map

$$\chi : \{\text{Coherent sheaves on } C\} \to \mathbb{Z}$$

(usually called *Euler characteristic*) satisfying the following axioms:

- $\chi(\kappa_P) = 1$ for every skyscraper sheaf κ_P supported at a point $P \in C(k)$. (The sheaf κ_P is defined as $\kappa_P := i_{P,*}(\kappa(P)^{\sim})$ where $i_P\colon \operatorname{Spec}(\kappa(P)) \to C$ is the natural map (a closed immersion), and $\kappa(P)^{\sim}$ is the quasi-coherent sheaf on $\operatorname{Spec}(\kappa(P))$ corresponding to the $\kappa(P)$-vector space $\kappa(P)$.)

- χ is additive in short exact sequences, i.e.: If there is a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0,$$

  of coherent sheaves on C, then $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$.

The goal of this exercise is to prove the *Riemann–Roch formula*

$$\chi(\mathscr{O}_C(D)) = \deg(D) + \chi(\mathscr{O}_C) \tag{0.1}$$

for every Cartier divisor D on C.

i) Let $P \in C(k)$ be a point and denote by κ_P the skyscraper sheaf supported at P. Show that there is a short exact sequence

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C(P) \longrightarrow \kappa_P \longrightarrow 0\,.$$

Deduce that for every Cartier D on C there is a short exact sequence

$$0 \longrightarrow \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C(D+P) \longrightarrow \kappa_P \longrightarrow 0\,.$$

ii) Prove Equation 0.1 for every effective divisor $D = \sum_{i=1}^{r} n_i[P_i]$, $n_i > 0$ by induction on $\deg(D)$.

iii) Use ii) to prove Equation 0.1 for every divisor by writing $D = D_1 - D_2$ as a difference of effective divisors and inducting on $\deg(D_2)$.

Remark: In the upcoming lectures we will define such a χ using sheaf cohomology: $\chi(\mathcal{F}) = \dim_k \mathrm{H}^0(C, \mathcal{F}) - \dim_k \mathrm{H}^1(C, \mathcal{F})$.